Lecture notes for

Financial Mathematics

Ingmar Schumacher

Version June 2026

Important:
The companion material to these lecture notes is the TD booklet. These lecture notes provide the main course material.

This is a course booklet that is still work in progress. For comments or if you find mistakes please contact me via email ingmar.schumacher@outlook.com.

Contents

What we do in this course 4

How you should study 5

Lecture 1 7

Brief revisions 7

Logarithms 9

Progressions - arithmetic 10

Progressions - geometric 12

Take-away box 16

Lecture 2 - simple interest 18

Take-away box 27

Lecture 3 - compound interest 28

Generic formula for compound interest 32

Continuous compounding 36

Take-away box 41

Lecture 4 - comparing interest rates 43

Take-away box 47

Lecture 5 - annuities **49**

Ordinary simple annuity 49

Annuity due 52

General annuity 53

(Net) Present Value 55

Internal Rate of Return 57

Take-away box 61

Lecture 6 - loan amortization **64**

Formula for loan amortization 65

Amortization table 67

Debt outstanding 69

Take-away box 71

Some final words **73**

> I think basic mathematics is really underrated. If you're going to make money, if you're going to invest money, your basic math should be really good. You don't need to learn geometry, trigonometry, calculus, or any of the complicated stuff if you're just going into business. But you want arithmetic, probability, and statistics. Those are extremely important. Crack open a basic math book, and make sure you are really good at multiplying, dividing, compounding, probability, and statistics.

(The Almanack of Naval Ravikant)

What we do in this course

Essentially this sentence is the key to understanding what we do in this course:

Time is money.

Imagine you need to choose between receiving 100€ now or in one year. What would be your choice? In this case it is reasonably easy - about everyone would choose to receive the 100€ immediately. There are many reasons for this choice. For example, you could place that money into a fixed term bank account and receive interest on this. In this case, after one year, you would have more than 100€. On the other hand, you may fear that in one year the 100€ would be worth much less due to inflation, or because you fear you may have died in the meantime. Clearly, having the money in the pocket right now carries less risk than the promise of the same money in the future.

What would, however, happen, if instead of 100€ in one year you would be offered 150€? Would you still prefer the 100€ today? Maybe, maybe not. If you prefer to receive 150€ in one year than 100€ now, then this implies you require an additional value 50% for waiting one year. We say that you apply an annual interest rate of 50%. If this rate is constant, then 100€ today would be equivalent to 200€ in two years for you.

If you want to compare money at different points in time, then you must first 'covert' it into the same point in time.

As another example you may think of inflation. If you have 300€ now, but there is an inflation rate of 10%, then in one year the 300€ will be worth less than today. We can calculate precisely by how much. In fact, at an inflation rate of 10%, the goods that would cost 272.73€ today would cost you 300€ in one year. In other words, 300€ in one year, at an interest rate of 10%, have the same value as 272.73€ today.

Hence you notice that there are three ingredients needed in order to compare money across time: it is the amount of money, the interest rate that gets applied, and the period of time that is of concern. There are different ways in which values can be calculated over time, but essentially this is all there is to it. Now that you understood the idea of the course we can revise some basics and then study the different ways in which money can be valued over time. And always remember: *time is money*.

How you should study

This will be a difficult course for some of you. Sometimes you might feel like leaving this aside and study everything later. However, it is absolutely essential that you do not procrastinate: After the lecture, go through the lecture notes and do the exercises. If you find out that some of these are difficult for you, then solve them with a colleague or contact me.

In my experience, many students lack a lot of the basics, such as equation solving, dealing with exponents, and especially translating a question into the language of mathematics and then translating the mathematical answer back into English. If you sit down and work on the exercises you will immediately see where you have problems. Do not give up but instead solve the problem, find the right answer and understand it. Do this by going back to the lecture notes, by searching for solutions online, and potentially talking to colleagues. But, first and foremost, this course is about giving *you* the skills to solve the problems that we address here, it is about providing *you* with an understanding of the concepts that we discuss. Thus, eventually the goal is that you are able to understand and solve the problems alone. Once you can do that, then you are well-prepared for the final exam and for the rest of the mathematics that you will face. Furthermore, the methods that you will learn here will prove to be useful whenever you will face investment projects in your personal or professional life, so do not hesitate to take advantage of the learning process that guides this course.

A word of advice: Each subsequent lecture builds upon the previous one. Thus, if you have trouble to understand a previous lecture, you will certainly be lost in the next one. Do not get in the position of wasting your time like this but instead study and go through the lecture notes after each course and do the exercises.

Lecture 1

Some important remarks:

- ☐ I am using the UK/US accounting way to write numbers, meaning a 'dot' is used for the decimal separator, while a 'comma' is used as a three-digit separator. Thus, the value 1000000 and 25 cents is written as 1,000,000.25.
- ☐ Multiplication can be written as $ab = a \times b = a \cdot b$.
- ☐ Days in a year according to convention (ordinary interest): 360
- ☐ Be careful about the starting and the end dates, e.g. there is a difference between 'beginning' and 'end' of year. Example: from beginning of 2019 to beginning of 2020 corresponds to one year; while from beginning of 2019 to end of 2020 corresponds to two years.
- ☐ Only round the final solution to the nearest cent (if necessary), do not round before. For x.xxy, round up if $y \geq 5$, round down if $y < 5$. Example: 1,003.045 becomes 1,003.05, while 1,003.043 becomes 1,003.04.

Brief revisions

$$x^a = \underbrace{x \cdot ... \cdot x}_{\text{a times}}$$

for $x \in \mathcal{R}$, $a \in \mathcal{R}$.

- ☐ x is the base, a the exponent (or power).
- ☐ we say the "a^{th} power of x"
- ☐ 2^{nd} power is square of x^2
- ☐ 3^{rd} power is cube of x^3

These are the four important rules that you have to remember:

- ☐ $x^m x^n = x^{m+n}$.
- ☐ $x^{-m} = \frac{1}{x^m}$.
- ☐ $(x^m)^n = x^{mn}$.
- ☐ $(xy)^m = x^m y^m$.

Reminder:

- ☐ $(x^m)^n \neq x^{m^n}$. Example: $x = 3$, $m = 2$, $n = 3$. Then $(x^m)^n = (3^2)^3 = (9)^3 = 729$. Instead, $x^{m^n} = 3^{2^3} = 3^8 = 6,561$.
- ☐ We should remember that $x^0 = 1$. Proof that $x^0 = 1$: $\frac{x^a}{x^a} = x^{a-a} = x^0$
- ☐ $x^{a/b} = \sqrt[b]{x^a}$

Special care if $x = 0$, as x^a is, if

- ☐ $a > 0 \implies 0^a = 0$
- ☐ $a = 0 \implies$ indeterminate or 1
- ☐ $a < 0 \implies 0^a = \infty$

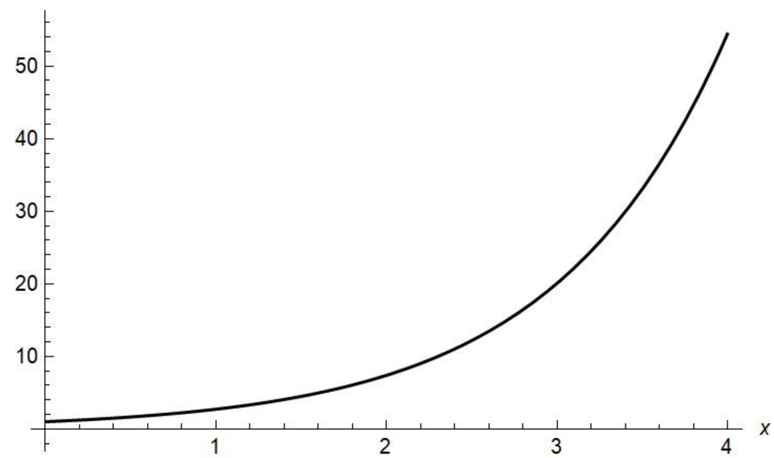
Examples:

- ☐ $(x/y)^b = x^b/y^b = y^{-b}/x^{-b} = (y/x)^{-b}$
- ☐ $27^{-2/3} = \left((27^2)^{1/3}\right)^{-1} = \left(\left((3^3)^{1/3}\right)^2\right)^{-1} = 1/9$
- ☐ Solve $10^{-2}(1 + x)^2 = 1$ for x. ($x = -11$ or $x = 9$)

Euler's number: One case of exponents that we are going to encounter a lot is the case where the base is e. Base e is also called Euler's number, or Napier's constant, with $e = \lim_{n\to\infty}(1 + 1/n)^n \approx 2.71828$. This will play an important role later in compound interest.

We have, as above, the following rules:

Figure 1: Example of exponential function



- ☐ $e^a e^b = e^{a+b}$
- ☐ $e^{-a} = \frac{1}{e^a}$.
- ☐ $(e^a)^b = e^{ab}$.
- ☐ $e^0 = 1$.

Logarithms

Sometimes we need to solve an equation, such as $x^a = y$, for the exponent a. This is not easy unless there are special cases or we iterate numerically. However, there is one tool, one definition, that we can use to solve for a, it's the *logarithm*. It's simply a definition.

$$x^a = y \iff a = \log_x y. \tag{1}$$

We say that a is the logarithm with base x of y.

Examples:

- ☐ $\log_2 16 = 4 \Longleftrightarrow 2^4 = 16$, where 2 is the base, 4 the exponent and 16 the required solution.
- ☐ $\log_8 19 = 1/2 \Longleftrightarrow 81^{1/2} = 9$
- ☐ Assume that $10 = (1 + 0.1)^t$. Solve for t. Then we use the definition of the logarithm, where 1.1 is our base, t the exponent, and obtain $t = \log_{1.1} 10$. Using the calculator, we then find $t = 24.1589$.

A special case is when the base is e, or Euler's number. In this case we would want to solve an equation such as $e^t = y$ for t. How do we do this? In the same way as before. We use the definition of the logarithm and get $t = \log_e y$. The logarithm with base e is, however, a special case, and we only need to remember that it is defined as the *natural logarithm*, where instead of $\log_e y$ we write $\ln y$.

Progressions - arithmetic

A progression is *arithmetic* if consecutive terms are separated by a *common difference.*

Examples

- ☐ 1, 2, 3...
- ☐ 23, 46, 69...

The basics:

- ☐ x_1 - the first term
- ☐ x_t - the t$^{\text{th}}$ term, for $t = 1, ..., T$
- ☐ d - the common difference

How we can find the t$^{\text{th}}$ term?

- ☐ sequence x_1, $x_2 = x_1 + d$, $x_3 = x_2 + d = x_1 + d + d = x_1 + 2d$, ...
- ☐ this gives us the **formula for the t$^{\text{th}}$ term of an arithmetic progression**

$$\boxed{x_t = x_1 + (t-1)d.} \tag{2}$$

How we can find the sum (S_t) of the x_t until t$^{\text{th}}$ term?

- ☐ first we write from front

$$S_t = x_1 + (x_1 + d) + (x_1 + 2d) + ... + (x_t - 2d) + (x_t - d) + x_t.$$

- ☐ then we write from back

$$S_t = x_t + (x_t - d) + (x_t - 2d) + ... + (x_1 + 2d) + (x_1 + d) + x_1.$$

- ☐ then we simply add the two

$$2S_t = (x_1 + x_t) + (x_1 + x_t) + ... + (x_1 + x_t) = t(x_1 + x_t).$$

- ☐ Finally, this gives us the **formula for the sum (S_t) of the x_t until t$^{\text{th}}$ term of an arithmetic progression**

$$\boxed{S_t = \tfrac{t}{2}(x_1 + x_t).} \tag{3}$$

Example: We have 1, 2, 3, 4, etc. Find the sum of the first 100 terms.

- ☐ *Important:* From now on we are going to use a standardized approach to answering questions. For each question you write down which approach you are using, the formulas required, the knowns and the unknowns, as well as your derivation and answers.

- ☐ Approach: arithmetic progression.
- ☐ Formulas: $x_t = x_1 + (t-1)d$ and $S_t = \frac{t}{2}(x_1 + x_t)$.
- ☐ Knowns: $x_1 = 1$, $d = 1$, $t = 100$. Unknowns: S_t and x_t.
- ☐ Derivation: first find x_{100}. This we do by solving $x_{100} = 1 + (100-1)1 = 100$. Then we substitute this into $S_{100} = (1+100)\frac{100}{2} = 5050$.
- ☐ Answer: The sum of the first 100 terms is $S_{100} = 5050$.

Progressions - geometric

A progression is *geometric* if consecutive terms are separated by a *common ratio*.

Examples

- ☐ 1, x, x^2. (first term 1, common ratio x)
- ☐ 10, 5, 2.5, 1.25. (first term 10, common ratio 1/2)

The basics:

- ☐ x_1 - the first term
- ☐ x_t - the t$^{\text{th}}$ term, for $t = 1, ..., T$
- ☐ a - the common ratio

How we can find the t$^{\text{th}}$ term?

- ☐ sequence x_1, ax_1, a^2x_1, ...

- ☐ this gives us the **formula for the t^th^ term of a geometric progression**

$$\boxed{x_t = x_1 a^{t-1}.} \tag{4}$$

How we can find the sum (S_n) of x_n until n$^{\text{th}}$ term?

- ☐ We know that the sum is

$$S_t = x_1 + ax_1 + a^2x_1 + ... + a^{t-2}x_1 + a^{t-1}x_1$$

- ☐ Multiply this sequence by a gives

$$aS_t = ax_1 + a^2x_1 + a^3x_1 + ... + a^{t-1}x_1 + a^tx_1$$

- ☐ Then we can calculate $S_t - aS_t = x_1 - a^tx_1$, which is equal to $(1-a)S_t = (1-a^t)x_1$.
- ☐ Finally this gives us the **formula for the sum (S_t) of x_t until t$^{\text{th}}$ term of a geometric progression**

$$\boxed{S_t = \frac{1-a^t}{1-a}x_1.} \tag{5}$$

Example: We have 1, b, b^2, b^3, etc. Find the sum of the first 10 terms.

- ☐ *Important:* Remember our standardized approach to answering questions.
- ☐ Approach: geometric progression.
- ☐ Formulas: $x_t = x_1 a^{t-1}$ and $S_t = \frac{1-a^t}{1-a}x_1$.
- ☐ Knowns: $x_1 = 1$, $a = b$, $N = 10$. Unknowns: S_t.
- ☐ Derivation: $S_{10} = \frac{1-b^{10}}{1-b}1$.
- ☐ Answer: The sum of the first 10 terms is $S_{10} = \frac{1-b^{10}}{1-b}$.

Assume now that $b = 2$, then $S_{10} = 2^{10} - 1 = 1023$.

Special case: $a = 1$, then $S_{10} = \frac{1-1^{10}}{1-1} = \frac{0}{0}$ (undefined). But we know from inspection that $\underbrace{1 + 1 + ... + 1}_{\text{10 times}} = 10$. This does not mean that the formula above is wrong, it simply cannot handle the case above as is. But if you use l'Hôpital's rule[1] on the formula above then it gives the correct solution.

The **infinite** geometric progression is a special case.

- ☐ An example is 1, 1/2, 1/4, 1/8, etc. Here $x = 1$ and $a = 1/2$. Then $S_t = x_1 \frac{1-a^t}{1-a}$ becomes $S_t = 1\frac{1-(1/2)^t}{1-1/2} = 2 - (1/2)^{t-1}$. For $t = \infty$ we then get $S_\infty = 2$.

- ☐ Remember: if $b \in (-1, 1)$, then as $t \to \infty$ we have $b^t \to 0$, implying that our formula for the infinite geometric progression is given by

$$S_\infty = \frac{x_1}{1-b}.$$

- ☐ Example 1: find the sum of 10, 1, 0.1, 0.01, ...
  Then $x_1 = 10$, $b = 1/10$, and $S_\infty = \frac{10}{1-1/10} = \frac{100}{9}$.

- ☐ Example 2: find the sum of $\frac{1}{1+x}$, $\frac{1}{(1+x)^2}$, $\frac{1}{(1+x)^3}$, etc.
  Then $x_1 = \frac{1}{1+x}$, and $b = \frac{1}{1+x}$, and thus $S_\infty = \frac{(1+x)^{-1}}{1-(1+x)^{-1}} = \frac{1}{1+x-1} = 1/x$.

Example:

- ☐ It's January and the value of your car is 30,000€ now. Its value though is reduced by 10% at the end of each year. What is its value going to be at the end of 10 years?

- ☐ Approach: geometric progression.

[1] l'Hôpital's rule says that, if $\lim_{x\to b} g(x) = \lim_{x\to b} f(x) = 0$ (or ∞), then $\lim_{x\to b} \frac{g(x)}{f(x)} = \lim_{x\to b} \frac{g'(x)}{f'(x)}$.

- ☐ Formulas: $x_t = x_1 a^{t-1}$.
- ☐ Knowns: $x_1 = 30,000$, $a = 0.9$, $t = 11$ (be careful with the dates). Unknowns: x_{11}.
- ☐ Derivation: $x_{11} = 30000x0.9^{11} = 9414.318$.
- ☐ Answer: The value of my car after 10 years is $S_{11} = 9,414.32$€.

Take-away box

- ☐ **Laws of Exponents**

  TAG: $\underbrace{x \times x \times ... \times x}_{m\ times} = x^m.$

  We should remember that $x^0 = 1$.

  - $x^m x^n = x^{m+n}$.
  - $x^{-m} = \frac{1}{x^m}$.
  - $(x^m)^n = x^{mn}$.
  - $(xy)^m = x^m y^m$.

- ☐ **Quadratic Formula**

  TAG: *Solving a Polynomial of order two.*

  Assume
  $$ax^2 + bx + c = 0,$$
  then use
  $$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- ☐ **Progressions (arithmetic)**

  TAG: *ADGR - **A**rithmetic **D**ifference and **G**eometric **R**atio.*

  - x_1 - first term
  - x_t - t$^{\text{th}}$ term
  - d - common difference
  - S_t - sum of first n terms

Finding the t$^{\text{th}}$ term:

$$x_t = x_1 + d(t-1) \tag{6}$$

Finding the sum S_t

$$S_t = \frac{t}{2}\Big(x_1 + x_t\Big) \tag{7}$$

☐ **Progressions (geometric)**

TAG: *ADGR* - ***A**rithmetic **D**ifference and **G**eometric **R**atio.*

- x_1 - first term
- x_t - t$^{\text{th}}$ term
- a - common ratio
- S_t - sum of first n terms

Finding the t$^{\text{th}}$ term:

$$x_t = a^{t-1}x_1 \tag{8}$$

Finding the sum S_t

$$S_t = \frac{a^t - 1}{a - 1}x_1 \tag{9}$$

In case of infinite geometric progressions use

$$S_\infty = \frac{x_1}{1 - a} \tag{10}$$

Lecture 2 - simple interest

If you borrow or lend money, or if you calculate values over time, then you need a way to compare current values with future or past values. For example, if you place 10,000€ into a bank account, then what is its value going to be in the one year? If you take out a short-term loan of 25,000€ for your company, say over a period of 30 days, then how much is the total cost of that loan?

The return that you get when you hold money in a bank account is called *interest*. We can calculate interest in several ways: one way is based on *simple* interest, the other is *compound* interest. When does which apply?

- ☐ Simple interest: short-term loans or biweekly mortgage; Certificates of deposit[2]; Simple interest tends to be used for periods less than one year.
- ☐ Compound interest: contracts with duration of more than one year; most deposits in bank accounts; long term contracts such as mortgages.

If a contract is based on **simple interest** then the interest is calculated on the *initial value*, what we also call the *principal* or the *capital invested*. At the end of the contract, the principal plus the interest becomes the *accumulated value*.

Notation:

- ☐ P - principal / present value of S / initial value / capital invested
- ☐ I - simple interest
- ☐ S - accumulated value
- ☐ r - interest rate (per year)

[2] A Certificate of deposit is a deposit in a bank for a fixed period of time, usually with a higher a interest rate.

☐ t - time in years

The accumulated value S is the principal P times the accumulation factor $(1 + rt)$ at simple interest I. The formula that we are going to use is given by

$$S = P + I = \underbrace{(1 + rt)}_{\text{accumulation factor}} P \tag{11}$$

The formula for the total interest payment is $I = rtP$.

Relation to arithmetic progression:

☐ Remember that the arithmetic progression is of form $x_t = x_1 + (t - 1)d$. Translated into the notation for simple interest we get $S_t = P + rtP$ (why not $(t - 1)$ as before?[3]).

Remarks:

☐ The interest rate, if not defined otherwise, is always given as the annual interest rate. If it is defined otherwise (you will learn this later), then you need to convert it into an annual interest rate.

☐ (**Beware**: common mistake!) Be careful with the interest rate. The interest rate can be given in percent (e.g. 3.5%), or in decimal points (e.g. 0.035). If there is no percentage sign (%) behind the number for the interest rate, then the interest rate is generally given in decimal points.

☐ Time is always in years. If it is not given in years then you need to convert it into years.
1 year $\widehat{=}$ 4 quarters (ex.: 3 quarters in years is $t = 3/4$)
1 year $\widehat{=}$ 12 months (ex.: 11 months in years is $t = 11/12$)

[3]When you place money into your bank, you do this at $t = 0$. After the first period you already get interest.

1 year $\widehat{=}$ 360 days (ex.: 301 days equals $t = 301/360$)
1 year $\widehat{=}$ 8640 hours (ex.: 24 hours equals $t = 24/8640$)

- ☐ 1 year 2 months and 3 days is $t = 1 + 2/12 + 3/360 = 1.175$
- ☐ We distinguish between exact simple interest and ordinary simple interest. Exact simple interest uses 365 days in a year, while ordinary simple interest uses approximate time, meaning each month has 30 days. A year thus has 360 days.

For the rest of the course we will use approximate time, thus ordinary simple interest, implying 30 days per months and 360 days in a year.

- ☐ Calculate *accumulated value* (principal + interest), solve for S (this is the formula I gave you)

$$S = P(1 + rt)$$

Calculate *principal*, solve for P

$$P = S/(1 + rt)$$

Calculate *rate of interest*, solve for r

$$r = (1/t)(S/P - 1)$$

Calculate *time*, solve for t

$$t = (1/r)(S/P - 1)$$

Example 1: I lend you 100€ at simple interest of 3% for 10 years. How much would you need to pay back?

- ☐ Approach: simple interest.
- ☐ Formula: $S = (1 + rt)P$.
- ☐ Knowns: $r = 3\% = 0.03$, $P = 100$, $t = 10$. Unknowns: S.

- ☐ Derivation: $S = (1 + 0.03 \cdot 10)100 = 130$.
- ☐ Answer: You will need to pay back 130€.

Example 2: You have 200€. You want to invest this for 3 years and hope to have 400€ afterwards. What is the required simple interest rate?

- ☐ Approach: simple interest.
- ☐ Formula: $S_t = (1 + rt)P$.
- ☐ Knowns: $S = 400$, $P = 200$, $t = 3$. Unknowns: r.
- ☐ Derivation: $400 = (1 + r \cdot 3)200$. Solving for r gives $r = 33\%$.
- ☐ Answer: You would need to get a simple interest rate of $r = 33\%$.

Example 3: You will have 200€ in your bank account in 180 days at a simple interest rate of 11%. How much money would you need to put into the account to have the required amount?

- ☐ Approach: simple interest.
- ☐ Formula: $S_t = (1 + rt)P$.
- ☐ Knowns: $S = 200$, $r = 0.11$, $t = 180/360$. Unknowns: P.
- ☐ Derivation: $200 = (1 + 0.11 \cdot 180/360)P$. Solving for P gives $P = 189.5735$.
- ☐ Answer: You would need to place a principal of $P = 189.57$€.

Example 4: You owe a drug lord quite some money. You have capital amounting to 34,000€. You want to invest this in the stock market in order to double this during the next 359 days. That's when you need to pay him back. What rate of return do you need to get on the stocks during those days?

- ☐ Approach: simple interest.

- ☐ Formula: $S_t = (1 + rt)P$.
- ☐ Knowns: $S = 2 \cdot 34000$, $P = 34000$, $t = 359/360$. Unknowns: r.
- ☐ Derivation: $2 = (1 + r \cdot 359/360)$. Solving for r gives $r = 360/359 = 1.002786$.
- ☐ Answer: You would need to get a rate of return amounting to $r = 100.28\%$.

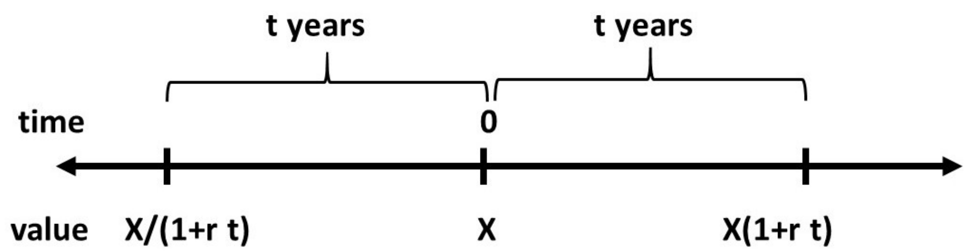
Example 5: You have principal P€. You want to double this during the next years. You will get a simple interest rate of 1%. How many years do you need to wait?

- ☐ Approach: simple interest.
- ☐ Formula: $S_t = (1 + rt)P$.
- ☐ Knowns: $S = 2P$, $P = P$, $r = 0.01$. Unknowns: t.
- ☐ Derivation: $2 = (1 + 0.01t)$. Solving for t gives $t = 100$.
- ☐ Answer: You would need to get wait $t = 100$ years.

Example 6: You have to invest 500,000€ now. In 10 years you will have to pay the money back. How much money will the total value of the loan be if the simple interest rate is 5%? How large is your interest payment?

- ☐ Approach: simple interest.
- ☐ Formulas: $S_t = (1 + rt)P$, $I = rtP$.
- ☐ Knowns: $P = 500,000$, $t = 10$, $r = 0.05$. Unknowns: S and I.
- ☐ Derivation: $S = (1 + 0.05 \cdot 10)500000$. Solving for S gives $S = 750,000$. $I = 0.05 \cdot 10 \cdot 500000 = 250,000$.
- ☐ Answer: You would need to pay interest of 250,000€ and the total value of the loan is 750,000€.

Figure 2: Drawing a timeline for aid



One issue that one needs to be careful with when values are calculated back and forth over time. In our formula, time cannot be negative. Here a diagram in the form of a timeline, such as Figure 2, may be helpful.

Answer questions related to changing values over time using the following steps:

Step 1 Draw a corresponding diagram (such as Figure 2)

Step 2 Choose a so-called *focal date* (one specific point in time that we use to compare values)

Step 3 Calculate/express all values at this chosen focal date

Important: your calculated values will vary slightly depending the chosen focal date.[4] Also, (**beware**) one common mistake is the following: assume you calculate the accumulated value between $t = 0$ and $t = t_1$, and also between t_1 and t_2, at an interest rate of a and with principal $P = 1$. Then a common mistake is that students first calculate the accumulated between $t = 0$ and $t = 1$, and then use the result as the principal for the period $t = 1$ and $t = 2$. If this is done then we obtain $S = (1 + at_1)(1 + at_2)$. However, this is wrong, because in this case

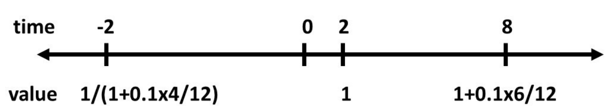
[4] This won't be the case any longer in the compound interest case and it only applies to the simple interest problems.

you calculate the interest not on the original principal but instead on the original principal and the interest that was obtained during that period. If you calculate the accumulated value correctly, directly between $t = 0$ and $t = t_2$, then you would have $S = (1 + a(t_1 + t_2))$. Comparing this to the wrong method, then the wrong method would then give $(1+at_1)(1+at_2) = 1+a(t_1+t_2)+a^2t_1t_2$, which is larger (by $a^2t_1t_2$) than $S = (1 + a(t_1 + t_2))$. As a is the interest rate, which tends to be small, and the t terms also tend to be less than one, then the factor $a^2t_1t_2$ is usually not a big amount. Sometimes, however, it is impossible to calculate the interest only on the original a principal (e.g. if a certain sum was repaid after a period of time), in which case we simply have to live with this approximation.

Example 7: Assume your salary in two months is 1€. Given a simple interest rate of 10%, how much would it be worth today, how much 2 months ago, and how much in 8 months?

- ☐ Approach: simple interest.
- ☐ Formula: $S_t = (1 + rt)P$. The way we define the knowns and unknowns

Figure 3: Drawing a timeline for aid



depends on from which point in time we calculate what. We calculate the value of the salary in 8 months from now. As the salary in 2 months from now is 1€, then between two months from now and 8 months from now we have 6 months.

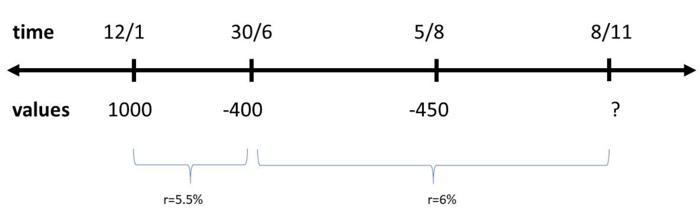
- ☐ Knowns: $P = 1$, $t = 6/12$, $r = 0.1$. Unknown: S.
- ☐ Derivation: $S = (1 + 0.1 \cdot 6/12)1$. Solving for S gives $S = 1.05$. Then we need to calculate the value of the salary today and two months ago, using time $t = 2/12$ as starting date. Let's start with today.

- ☐ Knowns: $S = 1$, $t = 2/12$, $r = 0.1$. Unknown: P.
- ☐ Derivation: $1 = (1 + 0.1 \cdot 2/12)P$. Solving for P gives $P = 0.9836066$. Then we finally calculate the value of the salary two months ago, remembering that $t = 2/12$ is our starting date.
- ☐ Knowns: $S = 1$, $t = 4/12$, $r = 0.1$. Unknown: P.
- ☐ Derivation: $1 = (1 + 0.1 \cdot 4/12)P$. Solving for P gives $P = 0.967742$.
- ☐ Answer: Using $t = 2/12$ as our focal date, the value of the salary two months ago is 0.97€, the value today is 0.98€, and the value in eight months from now is 1.05€.
- ☐ Remark: if we had used $t = 8/12$ as our starting date to e.g. calculate the value of the salary today, then we would have $S = 1.05$, $P = P$, $r = 0.1$ and $t = 8/12$. Substituting this into the formula for simple interest we get $1.05 = (1 + 0.1 \cdot 8/12)P$, and solving for P we obtain $P = 0.984375$. As you see, this value ($P = 0.9844$) is slightly larger than the one we calculated ($P = 0.9836$) when we used $t = 2/12$ as our starting date.

Example 8: Simon borrows 1000€ on the 12$^{\text{th}}$ of January at a simple interest rate of 5.5%. He repays 400€ on the 30$^{\text{th}}$ of June and 450€ on the 5$^{\text{th}}$ of August. From the 30$^{\text{th}}$ of June onwards the interest rate changes to 6%. What is the balance due on the 8$^{\text{th}}$ of November?

- ☐ Approach: simple interest.
- ☐ Formula: $S_t = (1 + rt)P$.

Figure 4: Drawing a timeline for aid



- ☐ Knowns: $P = 1000$, $r = 0.055$. Unknown: S. Several t. From 12$^{\text{th}}$ of January to 30$^{\text{th}}$ of June is 18+5·30 days, thus $t = 168/360$. From 30$^{\text{th}}$ of June to 5$^{\text{th}}$ of August is 35 days, thus $t = 35/360$. From 5$^{\text{th}}$ of August to 8$^{\text{th}}$ of November is 25+2·30+8 days, thus $t = 93/360$.

- ☐ Derivation: $S = \Big((1000(1+0.055\cdot168/360)-400)(1+0.055\cdot35/360)-450\Big)(1+0.06\cdot93/360)$. This gives $S = 181.787$.

- ☐ Answer: The final balance is $S = 181.79$€.

Take-away box

☐ **Simple (short-term) interest rate**

TAG: *A loan with interest on only the principal.*

- P - principal
- S - accumulated value
- r - interest rate
- t - time
- I - (total) interest

$$(1 + rt)P = S \tag{12}$$

$$rtP = I \tag{13}$$

- *exact simple interest* is calculated based on 365 days
- *ordinary simple interest* (most commonly used) uses 360 days

Lecture 3 - compound interest

In the previous lecture we discussed how to calculate values based on simple interest. While that approach is 'simple', as suggested in the name, it is mostly used for specific situations only. While it is also a good approximation to compound interest in the short-run, i.e. less than one year, it is a very bad approximation for periods longer than one year. Furthermore, most contracts and most banks as well as most financial tools nowadays rely on compound interest. Interestingly, this has not always been the case. Contracts, such as loans, that relied on compound interest were forbidden by Roman law as well as in other countries during similar periods of time.

If a contract is based on **compound interest** then the interest is calculate not on the initial value, as it is the case for simple interest, but it is calculated on the accumulated value at each point in time. In short, interest compounds if interest earns interest.

Notation:

- ☐ P - principal / present value of S / initial value / capital invested
- ☐ I - compounded interest
- ☐ S - accumulated value
- ☐ i - per period (of compounding) interest rate
- ☐ n - total periods

The accumulated value S is the principal P times the accumulation factor $((1+i)^n)$ at compound interest I. The formula that we are going to use, which is also called the *Fundamental Compound Interest Formula*, is given by

$$S = P + I = \underbrace{(1+i)^n}_{\text{accumulation factor}} P \tag{14}$$

The formula for the total interest payment is $I = ((1+i)^n - 1)P$.

How do we derive this? Assume you have P in your bank account. Then at $t = 0$, when you just placed the money into the account, then no interest was able to accumulate yet. Thus you still have $S_0 = (1+i)^0 P = P$. After one year, you receive interest for the first time, and thus you have $S_1 = (1+i)P$. Notice that, for the moment, the amount is the same as in the case of simple interest (why?). The reason is that you did not yet receive any interest on interest. After two years, you received interest on the amount of money that you had in the bank account at time $t = 1$. How much money did you have in your bank account at time $t = 1$? It was $(1+i)P$. If you receive interest on this, then you have at $t = 2$ an amount $S_2 = (1+i)(1+i)P = (1+i)^2 P$.

Notice also the difference to simple interest. With compound interest, you have $S_2 = (1+i)^2 P = (1+i)(1+i)P = (1+2i+i^2)P$, while with simple interest you would have $S = (1+i2)P$. The difference, after two periods, is thus i^2, which is the interest on the interest.

We can explain this in another way, too. At $t = 0$ you have $S_0 = P$. At $t = 1$ you have the money that you placed into the bank account, P, and the interest, iP, which makes $S_1 = P + iP = (1+i)P$. At $t = 2$, you have the money that you have in the bank, which is S_1, plus the interest that you earned on that money, which is iS_1, which together gives $S_2 = S_1 + iS_1 = (1+i)S_1 = (1+i)(1+i)P = (1+i)^2 P$.

Relation to geometric progression:

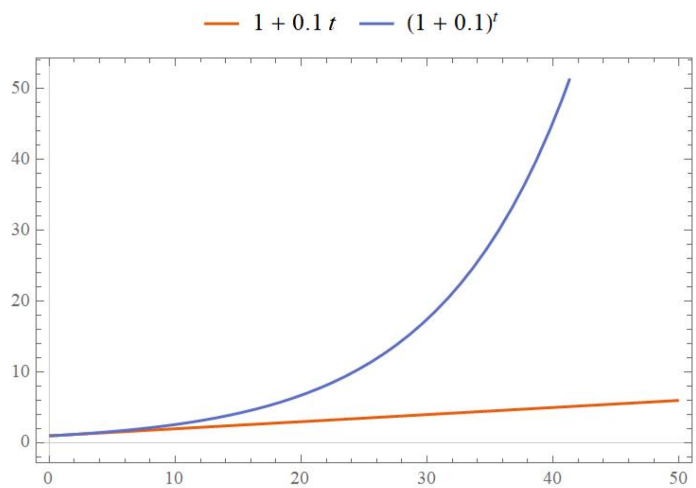
- Remember that the geometric progression is of form $x_t = x_1 a^{t-1}$. Translated into the notation for compound interest, we get $S_t = (1+i)^n P$ (why not $(n-1)$ as before?[5]), where $x_1 = P$, $x_t = S_t$ and $a = (1+i)$.

It is important to realize that the differences between simple and compound interest can be astonishing. Assume that $P = 1$ and $i = .1$. Then let's calculate the differences between these over time. Figure 5 gives some numerical solutions for the

[5] When you place money into your bank, you do this at $n = 0$. After the first period you already get interest.

accumulated values calculated based on the simple interest formula as well as the compound interest formula. One can see the fast and exponential increase in the accumulated value in the case of the compound interest case, compared to the linear increase in the accumulated value in the simple interest case.

Figure 5: Difference between simple and compound interest



n	simple	compound
0	1	1
1	1.1	1.1
2	1.2	1.21
5	1.5	1.61
10	2	2.59
50	6	117.39
100	11	13780

Remarks:

- ☐ The interest rate in the fundamental compound interest formula is always the per period rate. A period can be anything from yearly, quarterly, monthly, daily to even continuously. We will deal with a more general situation later.
- ☐ (**Beware**: common mistake) You need to remember the difference in the simple interest case and the compound interest case. The formula for the simple interest case multiplies r with t, whereas the formula for the compound interest case raises $(1 + i)$ to the power of n, a substantial difference!
- ☐ (**Beware**: common mistake) Be careful and check whether the period of the interest rate is the same as the compound frequency. If not, then you have to use a more general formula, see below. In the fundamental compound formula the interest rate is at the same periodicity as the compound frequency.

- ☐ Calculate the accumulated values by solving for S_n (this is the formula I gave you)

  $$S_n = (1+i)^n P.$$

  Calculate the principal, you solve for P

  $$P = \frac{S_n}{(1+i)^n}.$$

  Calculate the time by solving for n

  $$n = \log_{1+i}(S_n/P).$$

  Calculate the interest rate you solve for i

  $$i = (S_n/P)^{1/n} - 1.$$

Example 1: I lend you 100€ at an annual interest rate of 3% for 10 years. How much would you need to pay back?

- ☐ Approach: compound interest.
- ☐ Formula: $S = (1+i)^n P$.
- ☐ Knowns: $i = 3\% = 0.03$, $P = 100$, $n = 10$. Unknowns: S.
- ☐ Derivation: $S = (1+0.03)^{10}100 = 134.392$.
- ☐ Answer: You will need to pay back 134.39€.

Example 2a: You have 200€. You want to invest this for 3 years and hope to have 400€ afterwards. What is the required per period interest rate?

- ☐ Approach: compound interest.
- ☐ Formula: $S_n = (1+i)^n P$.
- ☐ Knowns: $S = 400$, $P = 200$, $n = 3$. Unknowns: i.
- ☐ Derivation: $400 = (1+i)^3 200$. Then $2^{1/3} - 1 = i$. Solving for i gives $i = 0.259921\%$.

☐ Answer: You would need to get an annual interest rate of $i = 26\%$.

Example 2b: You have 200€. You want to invest this for 3 days and hope to have 400€ afterwards. What is the required per period interest rate?

☐ Approach: compound interest.

☐ Formula: $S_n = (1+i)^n P$.

☐ Knowns: $S = 400$, $P = 200$, $n = 3$. Unknowns: i.

☐ Derivation: $400 = (1+i)^3 200$. Then $2^{1/3} - 1 = i$. Solving for i gives $i = 0.259921\%$.

☐ Answer: You would need to get a daily interest rate of $i = 26\%$.

Generic formula for compound interest

As you see in Examples 2a and 2b, a similar question, though with different lengths of periods, leads to seemingly different interest rates. This is because these are per period interest rates, and the length of the periods was differently. Up to now we only worked with annual interest rates, in the case of simple discounting, or per period interest rates for compound interest. But this is, generally, not the case. In fact, most banks tend to advertise interest rates not only at different interest periods, such as monthly or daily interest rates, but also at different compound frequencies. It gets a bit more complicated when the period of the interest rate is not the same as the compound frequency.

A lesson from times past In the year 1683 a mathematician called Jacob Bernoulli worked on a particular problem:

> A bank account starts with 1€ and you receive an annual interest rate of 100 percent. If the interest is compounded only once, then the money in the bank account at the end of one year will be 2€. Jacob Bernoulli then asked the question: What happens if interest is received more frequently throughout the year?

Assume that interest is received twice in the year, then this implies that the semi-annual interest rate will be 50 percent ($i = j_2/2$, where $i = 1/2$). In this case the initial bank balance, which was 1€, received 50 percent interest in the first half of the year, and based on that amount another 50 percent in the second half of the year. This gives $(1+1/2)^2 = 2.25$. If interest is obtained on a quarterly basis, then we have $(1 + 1/4)^4 = 2.441$, and interest compounded monthly yields $(1 + 1/12)^{12} = 2.613$ Assuming that the frequency of compounding is n gives us a per period interest rate of 100%/n while the accumulation factor becomes $(1 + 1/n)^n$. Jacob Bernoulli then calculated what happens when n becomes very large, and thus the frequency of compounding very small. He noticed that weekly compounding, thus if $n = 52$, gives an accumulation factor of 2.693, and daily compounding gives an accumulation factor of 2.715. When n becomes very large, meaning that we compound interest at every moment in time, then the accumulation factor approaches 2.718. This number is what we call e, the Euler's number.

Notation:

- ☐ t - years
- ☐ n - number of interest periods
- ☐ m - frequency of compounding
- ☐ j_m - nominal annual interest rate compounded m-times per year (also called *Annual Percentage Rate*)
- ☐ i - per interest period interest rate (or proportional rate)
- ☐ r - interest rate with periodicity v

Then we have the following definitions.

$$n = m \cdot t,$$

and

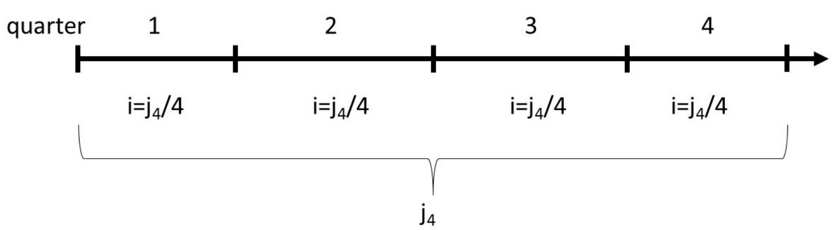
$$i = j_m/m,$$

and

$$j_m = vr.$$

As an example, j_4 is the nominal annual interest rate compounded 4-times a year,

Figure 6: $i = j_1/4$ is the quarterly interest rate



and thus $i = j_4/4$ is the quarterly interest rate, or the per period (in this case 4 periods, each lasting one quarter) interest rate. There are obviously many combinations at which interest rates can be provided. You can have, as examples, an annually compounded annual interest rate (j_1), a quarterly compounded annual interest rate (j_4), a daily compounded annual interest rate (j_{360}). You can also have a monthly compounded quarterly interest rate ($4r = j_{12}$ and thus $i = 4r/12$), a daily compounded daily interest rate ($i = j_{360}/360$), or a daily compounded monthly interest rate ($12r = j_{360}$ and thus $i = 12r/360$).

If we substitute $n = m \cdot t$ and $i = j_m/m$ into the Fundamental compound interest formula, then this gives us the complete **formula for compound interest rates** at generic compound frequencies and time periods:

$$\boxed{S = \left((1 + \tfrac{j_m}{m})^m\right)^t P} \tag{15}$$

This is the formula that you will have to learn. So what is the difference between this more general formula and the Fundamental formula? Let's take a look. We have $S = \left(1 + \frac{j_m}{m}\right)^{mt} P$, and we know that $i = j_m/m$ and $n = mt$. Then we can substitute these two relations into the general formula and we get $S = (1+i)^n P$, which is precisely the Fundamental formula. If you want to use the Fundamental formula, then you need to have the per period interest rate (i) and the total number of periods at which interest gets compounded (n). As we know that interest gets compounded m times a year, and our contract lasts for t years, then we know that in total there are $n = mt$ compounding periods. Furthermore, since we know that there are m compounding periods in a year, and in each period we receive interest of i, then we can use this to calculate the interest rate that applies throughout the whole year (the annual interest rate), which will be given by $mi = j_m$.

Example A: Assume you have 1000€ and you can invest it at a quarterly interest rate of 1% and the compound frequency is monthly. What is your accumulated value after 4 years?

- ☐ Approach: compound interest.
- ☐ Formula: $S = (1 + j_m/m)^{mt} P$ or $S = (1+i)^n P$, (You only need to use one of these two formulas to answer the question. I am showing both as an example.) as well as $rv = j_m$.
- ☐ Knowns: $P = 1000$, $m = 12$, $t = 4$, $r = 0.01$, $v = 4$. Unknowns: S, j_m.
- ☐ Derivation: Assume we use $S = (1 + j_m/m)^{mt} P$ to solve this problem. Then we need to find j_m first. We know that $rv = j_m$ and thus substituting we find a nominal annual rate of $j_{12} = 4 \cdot 0.01$. Then substituting everything in our formula we obtain $S = (1 + 0.04/12)^{12 \cdot 4} 1000 = 1173.2$. The other option is that we use the Fundamental formula $S = (1+i)^n P$. We know that $i = j_m/m = vr/m$, thus $i = 4/120.01 = 0.00333333$. Also, $mt = n$, thus $4 \cdot 12 = n$. Then using this in the formula we get $S = (1 + 0.00333333)^{48} 1000$, giving $S = 1173.2$.
- ☐ Answer: You would obtain an accumulated value of $S = 1173.2$€.

Continuous compounding

We said above that the compound frequency can be, for example, annual, monthly, quarterly, daily. It can also be much faster than this, namely compounding can be done on a continuous basis. The **formula**[6] **for continuous compounding** is thus

$$\boxed{S = e^{\delta t} P,} \tag{16}$$

where $\delta \equiv j_\infty$. We call δ also the *force of interest*. Remember that δ is the annual interest rate compounded infinitely many times. Now we are in a position to solve for other variables.

- ☐ Calculate the accumulated values by solving for S_t (this is the formula I gave you)

  $$S = \left(\left(1 + \frac{j_m}{m}\right)^m\right)^t P.$$

  Calculate the principal, you solve for P

  $$P = S\left(1 + \frac{j_m}{m}\right)^{-mt}.$$

  Calculate the time by solving for t

  $$t = \frac{1}{m} \log_{(1+\frac{j_m}{m})}(S/P).$$

  Calculate the interest rate you solve for j_m

  $$j_m = m\Big((S/P)^{\frac{1}{mt}} - 1\Big).$$

- ☐ For continuous compounding we have the accumulated value given by

  $$S = e^{\delta t} P,$$

  the principal can be obtained via

  $$P = S e^{-\delta t},$$

[6] We obtain this formula when we take the limit $\lim_{m\to\infty}(1 + j_m/m)^m = e^{j_\infty}$.

the time can be obtained through

$$t = \frac{1}{\delta}\ln(S/P),$$

and the interest rate δ we calculate based on

$$\delta = \frac{1}{t}\ln(S/P).$$

Solving for m is a problem in this equation and we cannot do this explicitly. There is an iterative approach, though, which will be shown later in the course (in the section on Internal Rate of Return).

Example 3: You will have 200€ in your bank account in 180 days at an annually compounded annual interest rate of 11%. How much money would you need to put into the account to have the required amount?

- ☐ Approach: compound interest.
- ☐ Formula: $S_t = (1 + j_m/m)^{mt}P$.
- ☐ Knowns: $S = 200$, $j_m = 0.11$, $m = 1$, $t = 180/360$. Unknowns: P.
- ☐ Derivation: $200 = (1 + 0.11)^{180/360}P$. Solving for P gives $P = 189.832$.
- ☐ Answer: You would need to place a principal of $P = 189.83$€.

Example 4: You owe a drug lord quite some money. You have capital amounting to 34,000€. You want to invest this in the stock market in order to double this during the next 359 days. That's when you need to pay him back. What annually compounded annual rate of return do you need to get on the stocks during those days?

- ☐ Approach: Compound interest.
- ☐ Formula: $S_t = (1 + j_m/m)^{mt}P$.
- ☐ Knowns: $S = 2 \cdot 34000$, $P = 34000$, $m = 1$, $t = 359/360$. Unknowns: j_m.

- ☐ Derivation: $2 = (1+j_1)^{359/360}$. Solving for $j_1 = 2^{360/359} - 1$ gives $j_1 = 360/359 = 1.00387$.

- ☐ Answer: You would need to get a rate of return amounting to $i = 100.39\%$.

Example 5: You have principal P€. You want to double this during the next years. You will get an annually compounded annual interest rate of 1%. How many years do you need to wait?

- ☐ Approach: compound interest.

- ☐ Formula: $S_t = (1 + j_m/m)^{mt} P$.

- ☐ Knowns: $S = 2P$, $P = P$, $j_m = 0.01$, $m = 1$. Unknowns: t.

- ☐ Derivation: $2 = (1 + 0.01)^t$. Solving for $t = \log_{1.01}(2)$ gives $t = 69.6607$.

- ☐ Answer: You would need to get wait 69 years and $\frac{66}{100} 360 = 237.6$ days.

Example 6: You have to invest 500,000€ now. In 10 years you will have to pay the money back. How much money will the total value of the loan be if the annually compounded, annual interest rate is 5%? How large is your interest payment?

- ☐ Approach: compound interest.

- ☐ Formulas: $S_t = (1 + j_m/m)^{mt} P$, $I = S - P$.

- ☐ Knowns: $P = 500,000$, $t = 10$, $j_m = 0.05$, $m = 1$. Unknowns: S and I.

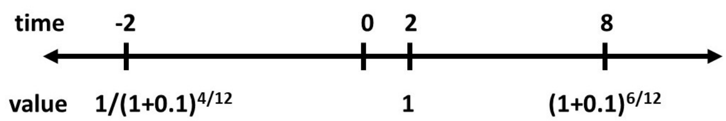
- ☐ Derivation: $S = (1 + 0.05)^{10} 500000$. Solving for S gives $S = 814,447$. $I = 814447 - 500000 = 314447$.

- ☐ Answer: You would need to pay interest of 314,447€ and the total value of the loan is 814,447€.

Example 7: Assume your salary in two months is 1€. Given an annually compounded annual interest rate of 10%, how much would it be worth today, how much 2 months ago, and how much in 8 months?

- ☐ Approach: compound interest.
- ☐ Formula: $S_t = (1 + j_m/m)^{mt}P$. The way we define the knowns and unknowns

Figure 7: Drawing a timeline for aid



depends on from which point in time we calculate what. We calculate the value of the salary in 8 months from now. As the salary in 2 months from now is 1€, then between two months from now and 8 months from now we have 6 months.

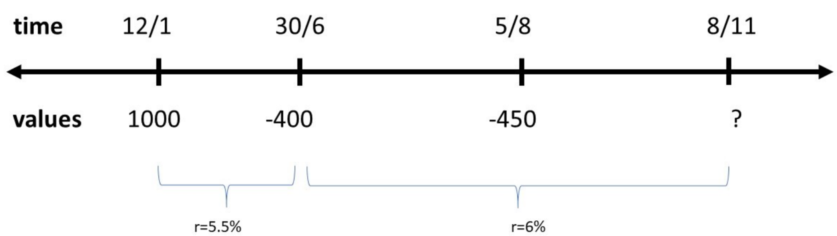
- ☐ Knowns: $P = 1$, $t = 6/12$, $j_m = 0.1$, $m = 1$. Unknown: S.
- ☐ Derivation: $S = (1 + 0.1)^{6/12}1$. Solving for S gives $S = 1.04881$. Then we need to calculate the value of the salary today and two months ago, using time $t = 2/12$ as starting date. Let's start with today.
- ☐ Knowns: $S = 1$, $t = 2/12$, $j_1 = 0.1$. Unknown: P.
- ☐ Derivation: $1 = (1 + 0.1)^{2/12}P$. Solving for P gives $P = 0.98424$. Then we finally calculate the value of the salary two months ago, remembering that $n = 2/12$ is our starting date.
- ☐ Knowns: $S = 1$, $t = 4/12$, $j_1 = 0.1$. Unknown: P.
- ☐ Derivation: $1 = (1 + 0.1)^{4/12}P$. Solving for P gives $P = 0.968729$.
- ☐ Answer: The value of the salary two months ago is 0.97€, the value today is 0.98€, and the value in eight months from now is 1.05€.

Example 8: Simon borrows 1000€ on the 12$^{\text{th}}$ of January at a compound interest rate of 5.5%. He repays 400€ on the 30$^{\text{th}}$ of June and 450€ on the 5$^{\text{th}}$ of August. From the 30$^{\text{th}}$ of June onwards the interest rate changes to 6%. What is the balance due on the 8$^{\text{th}}$ of November?

☐ Approach: compound interest.

☐ Formula: $S_t = (1 + j_m/m)^{mt} P$

Figure 8: Drawing a timeline for aid



☐ Knowns: $P = 1000$, $j_m = 0.055$, $m = 1$. Unknown: S. Several t. From 12$^{\text{th}}$ of January to 30$^{\text{th}}$ of June is 18+5·30 days, thus $t = 168/360$. From 30$^{\text{th}}$ of June to 5$^{\text{th}}$ of August is 35 days, thus $t = 35/360$. From 5$^{\text{th}}$ of August to 8$^{\text{th}}$ of November is 25+2·30+8 days, thus $t = 93/360$.

☐ Derivation: $S = \Big((1000(1 + 0.055)^{168/360} - 400)(1 + 0.055)^{35/360} - 450\Big)(1 + 0.06)^{93/360}$. This gives $S = 181.272$.

☐ Answer: The final balance is $S = 181.27$€.

Take-away box

☐ **Compound (long-term) interest rate**

TAG: *A loan with interest on interest.*

- P - principal
- S - accumulated value
- r - annual interest rate
- t - time

This formula applies for *annual compounding* only:

$$S = (1 + r)^t P \tag{17}$$

Let's be more precise, and this is the formula you should know (you can derive eq. (17) from this one):

- P - principal
- S - accumulated value
- j_m - annual interest rate
- m - compound frequency
- t - time

$$S = \left(1 + \frac{j_m}{m}\right)^{mt} P \tag{18}$$

☐ **Continuous compounding**

TAG: *A loan with interest on interest where interest compounds continuously.*

- P - principal
- S - accumulated value

- t - number of years
- δ - the interest rate (continuously compounded)
- i_n - per *interest* period interest rate

We define $\delta \equiv i_\infty$. Then

$$S = \lim_{n\to\infty} \left(\left(1 + \frac{i_n}{n}\right)^n\right)^t P = e^{\delta t} P. \tag{19}$$

Lecture 4 - comparing interest rates

In order to be able to compare interest rates we need a bit more in terms of definitions.

Notation:

- ☐ m_1 - frequency of compounding
- ☐ m_2 - (another) frequency of compounding
- ☐ j_m - nominal annual interest rate compounded m-times per year
- ☐ i - per interest period interest rate (or proportional rate)
- ☐ r - an interest rate with periodicity v
- ☐ j_e - annual effective (or equivalent) rate

We know that the per interest period interest rate i can be easily converted to an annual interest rate by using

$$i = j_m/m.$$

We call i also the *proportional rate* because it is the rate which is proportional to the annual one. For example, if $i = 0.01$ is the monthly interest rate (compounded monthly), then the nominal annual rate is $j_{12} = 12 \cdot 0.01$.

We also know that an interest rate with any periodicity v can be converted into a nominal annual rate by using

$$j_m = vr.$$

For example, if you obtain a quarterly interest rate of 3% compounded weekly, then this corresponds to a nominal annual rate of $j_{52} = 4 \cdot 0.03$, and to a per period interest rate (remember: per period means per period of compounding) of $i = vr/m$, which in this case is $i = 4 \cdot 0.01/52$.

Then there is a difference between a nominal rate j_m and an effective rate j_e. The annual interest rate that you are typically given by banks for e.g. loans or investments is the nominal interest rate j_m. This nominal interest rate is the rate before compounding has been taken into account. In contrast, the **effect**ive interest rate j_e is the interest rate that includes the **effect** of compounding. The effective interest rate allows us to easily compare different investment profiles. To be precise, the effective interest rate, also called the annual equivalent rate, is an annual interest rate compounded annually.

Imagine you have two investment profiles, both have the same principal, the same time horizon, and the same accumulated value. They only differ in the interest rates and the frequency of compounding. Let's assume one investment profile then is given by $S = (1 + j_{m_1}/m_1)^{m_1 t}P$, while the other investment profile is given by $S = (1 + j_{m_2}/m_2)^{m_2 t}P$. Since both profiles have the same accumulated value, then we can equate these two

$$(1 + j_{m_2}/m_2)^{m_2 t}P = S = (1 + j_{m_1}/m_1)^{m_1 t}P.$$

As both have the sample principal and the same time horizon, then we can simplify (divide by P and raise both sides to the power of $1/t$), and we get what is called the **Formula for Equivalent rates**:

$$\boxed{(1 + j_{m_2}/m_2)^{m_2} = (1 + j_{m_1}/m_1)^{m_1}.} \tag{20}$$

Using this formula, we can now find the nominal interest rate at one compound frequency that is equivalent (i.e. it gives rise to the same investment profile) to another nominal interest rate at another compound frequency. A similar formula can be used to convert to and from discrete compound frequencies to continuous compound frequencies. This formula[7] is

$$\delta = m \ln(1 + j_m/m).$$

Furthermore, and much simpler, as the annual equivalent rate is an annual interest rate that is compounded annually, then, using the formula above, we can simplify ($m_2 = 1$ and $r_e = j_{m_1}$) to obtain the formula for calculating the **annual equivalent rate**:

$$j_e = (1 + j_m/m)^m - 1,$$

[7] We derive this by equating $S = e^{\delta t}P$ with $S = (1 + j_m/m)^{mt}P$, which gives after diving by P and raising everything to the power of $1/t$ the equation $e^{\delta} = (1 + j_m/m)^m$. Then taking logs gives the equation as shown below.

and for continuous compounding we get

$$j_e = e^{\delta} - 1.$$

As you see, the formulas for the annual equivalent rates are only special cases of the equivalent rate formula equation (20), where we have $m_2 = 1$ and $j_e = j_{m_2}$.

Example 1: Find the annual equivalent rate for a 7% interest rate compounded a) monthly, b) daily, c) continuously.

- ☐ Approach: Annual equivalent interest rates.
- ☐ Formula: $j_e = (1 + j_m/m)^m - 1$ or $j_e = e^{\delta} - 1$.
- ☐ Knowns: $j_m = \delta = 0.07$, a) $m = 12$, b) $m = 360$ and c) $m = \infty$. Unknown: j_e.
- ☐ Derivation: a) $j_e = (1+0.07/12)^{12} - 1 = 0.0722901$, b) $j_e = (1+0.07/360)^{360} - 1 = 0.0725009$, c) $j_e = e^{0.07} - 1 = 0.0725082$.
- ☐ Answer: The annual equivalent rate is a) 7.23%, b) 7.25%, c) 7.251%.

Example 2: Find the annual equivalent rate for a 3% semi-annual interest rate compounded quarterly.

- ☐ Approach: Annual equivalent interest rates.
- ☐ Formula: $j_e = (1 + j_m/m)^m - 1$ and $vr = j_m$.
- ☐ Knowns: $r = 0.03$, $v = 2$, $m = 4$. Unknown: j_e, j_m.
- ☐ Derivation: $2 \cdot 0.03 = 0.06 = j_m$. Then $j_e = (1 + 0.06/4)^4 - 1 = 0.0613636$.
- ☐ Answer: The annual equivalent rate is 6.14%.

Example 3: Assume the annual interest rate 5% is compounded semi-annually. What is the equivalent rate if we compound a) daily, b) quarterly, c) annually, d) continuously?

☐ Approach: equivalent interest rates.

☐ Formula: $(1 + j_{m_2}/m_2)^{m_2} = (1 + j_{m_1}/m_1)^{m_1}$ and $1 + e^{\delta} = (1 + j_{m_1}/m_1)^{m_1}$.

☐ Knowns: $j_{m_2} = 0.05$, $m_2 = 2$. Unknown: j_{m_1}, a) $m_1 = 360$, b) $m_1 = 4$, c) $m_1 = 1$, d) $m_1 = \infty$.

☐ Derivation: We only derive this for case a): $(1 + 0.05/2)^2 = (1 + j_{m_1}/360)^{360}$, which gives us $j_{m_1} = 0.0493886$.

☐ Answer: The annual interest rate compounded daily that is equivalent to an annual interest rate of 5% compounded semi-annually is 4.94%.

Take-away box

- ☐ **Convert periodicity of nominal interest rates**
  (per period: e.g. semi-annual/quarterly/ monthly/daily rates)

  TAG: *The nominal rate* proportional *to a rate with another periodicity.*

  - j_m - nominal annual *interest* rate at compound frequency m
  - r - interest rate at periodicity v

$$j_m = v \times r \tag{21}$$

- ☐ **Equivalent rates**

  TAG: *What per period interest rate is equal to another interest rate with another periodicity?*

  - i_n - per period interest rate with n periods
  - j_m - per period interest rate with m periods

$$\left(1 + \frac{j_m}{m}\right)^m = \left(1 + \frac{i_n}{n}\right)^n \tag{22}$$

Given $\lim_{n\to\infty} \left(1 + \frac{i_n}{n}\right)^n = e^{i_n}$, define $\delta \equiv i_n$, then

$$\left(1 + \frac{j_m}{m}\right)^m = e^{\delta} \tag{23}$$

- ☐ **Annual equivalent rates**

  TAG: *What per period interest rate is equal to the* effective *interest rate?*

  - j_m - per period interest rate with n periods
  - j_e - annual equivalent interest rate

$$\left(1+\frac{j_m}{m}\right)^m = 1 + j_e \tag{24}$$

Given $\lim_{n\to\infty}\left(1+\frac{i_n}{n}\right)^n = e^{i_n}$, define $\delta \equiv i_n$, then

$$1 + j_e = e^{\delta} \tag{25}$$

Lecture 5 - annuities

Up to now we have studied what happens if we invest, or borrow, money at a specific point in time. However, in reality we do many transactions and often in regular intervals. For example, we tend to receive a monthly income, or we pay rent in monthly intervals, or for each birthday we get a financial boost from the family. It gets very cumbersome to use the formulas that we have in order to track the impact of these regular transactions, especially if we have many of them. Fortunately mathematics provides us with a neat solution for this in the form of a formula for annuities. This allows us to calculate how much money we would have in the future after having undertaken a series of payments at the end of periods. For example, we use annuities in order to calculate the total cost of a loan.

An annuity is a sequence of payments made at regular intervals in time.

We distinguish between three types of annuities - an ordinary simple annuity, an annuity due, and a general annuity.

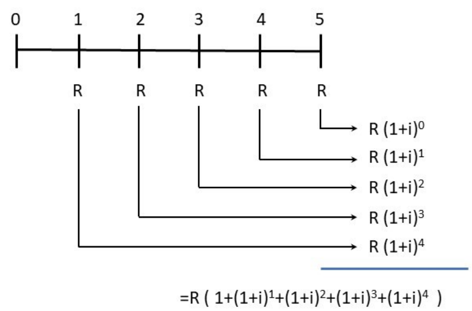
Ordinary simple annuity

Let us see what an ordinary simple annuity is. We start, as usual, with some notation.

Notation:

- ☐ R - regular (periodic) payment of the annuity
- ☐ n - number of payments (or number of periods)
- ☐ S - accumulated value
- ☐ i - per period interest rate

Figure 9: Example of an annuity with $n = 5$.



Let's derive the formula. If we receive income R at the end of the first period and we invest this for n periods at rate i, then at the end of the last period we would have $R(1+i)^{n-1}$.[8] If we receive income again at the end of the second period and we invest it, then we will also receive interest on this, albeit for one period less, $R(1+i)^{n-2}$. Income that we receive in the period before the last period receives interest only once, so that we would have $R(1+i)$, while the income that we receive in the last period gets no interest and is simply R. If we thus sum, from the last period to the first period, all payments, then we would get

$$S_n = R + R(1+i) + ... + R(1+i)^{n-2} + R(1+i)^{n-1}.$$

If you remember our progressions, then this is a geometric progression, with $x_1 = R$ and common ratio $a = (1+i)$. Using our formula for the geometric progression, which is $S_n = \frac{1-a^n}{1-a}x_1$, then we get the **formula for ordinary simple annuities**:

$$S = \frac{(1+i)^n - 1}{i}R. \tag{26}$$

[8] Be careful with the exponent, it should be $n-1$ as we receive the income at the end of the first period and thus, if we invest for two periods, then we only receive interest once on this, namely at the end of the second period.

An annuity is a *simple annuity* if payments are undertaken at the end of the interest period. Furthermore, an annuity is an *ordinary simple annuities* if the compounding period corresponds to the periodic payment period.

What one has to understand is that an annuity is simply a sequence of compound interest investments where the investments come at regular intervals in time and each investment is the same size (like a monthly rent payment). The formula for the annuity allows us to more easily solve this kind of sequence of compound investments. As an example, assume that at the end of each month we save 1000 EUR from our salary that we place into a bank account. The bank account pays a monthly compounded, monthly interest rate of 1%. We do this for three months. Then at the end of the first period we save 1000 EUR, and it is kept in the bank account until the end of period three, which gives it two months interest. Thus we calculate $S = 1000(1 + 0.01)^2 = 1020.1$. Then at the end of the second period we place 1000 EUR into the bank account, and it will give us one month interest (until the end of period three). Thus we get $S = 1000(1 + 0.01) = 1010$. At the end of period three we place 1000 EUR into the bank account again, and at that point we now evaluate the total amount of money in the account. This is going to be $1020.1 + 1010 + 1000 = 3030.1$. Let us, instead use our formula for the annuity. Then we have $S = 1000((1 + 0.01)^3 - 1)/0.01 = 3030.1$. So both methods give the same result (obviously...).

Example 1: You save 1000€ at the end of every year for 10 years at an annually compounded interest rate of 10%. What is the accumulated value?

- ☐ Approach: ordinary simple annuity (as the payment is done at the end of every period and the compound frequency corresponds to the payment frequency).
- ☐ Formula: $S = \frac{(1+i)^n - 1}{i} R$.
- ☐ Knowns: $R = 1000$, $n = 10$, $i = 0.1$. Unknown: S
- ☐ Derivation: $S = \frac{(1+0.1)^{10} - 1}{0.1} 1000$. This gives $S = 15937$.
- ☐ Answer: The final balance is $S = 15,937$€.

Annuity due

An annuity is an *annuity due* if payments are undertaken at the beginning of each period.

This is, for example, the case for rent payments, where rent tends to be paid at the beginning of a month. In this case, since payments are undertaken at the beginning of each period, then each payment receives an additional period of interest, and thus our **formula for annuity due** is

$$S = (1+i)\frac{(1+i)^n - 1}{i}R.$$

Otherwise it is the same as an ordinary simple annuity, in that the frequency of period payments corresponds to the frequency of compounding.

Example 2: In your bank account you have an accumulated value of 159,374.25€. You undertook a the periodic payment at the end of every year for 10 years. What is the amount of this periodic payment? What would be the periodic payment if you had undertaken the payments at the beginning of each year?

- ☐ Approach: ordinary simple annuity in the case where the payment is done at the end of every period (and we also know that the compound frequency corresponds to the payment frequency). Annuity due in the case where the payment is done at the beginning of every period

- ☐ Formula: ordinary simple annuity: $S = \frac{(1+i)^n-1}{i}R$. and annuity due: $S = (1+i)\frac{(1+i)^n-1}{i}R$.

- ☐ Knowns: $S = 159374.25$, $n = 10$, $i = 0.1$. Unknown: R

- ☐ Derivation: For the ordinary simple annuity: $159374.25 = \frac{(1+0.1)^{10}-1}{0.1}R$. This gives $R = 10000$. In the case of the annuity due we simply calculate $R = 10000/1.1$ (do you know why?).

- ☐ Answer: The periodic payment is $R = 10,000$€ for the case of the ordinary simple annuity, and $R = 10000/1.1$ for the annuity due.

In the previous example you see that, as in the case of the annuity due the payment is done at the beginning of each period, then the implication is that each payment receives an additional times interest. Thus, if the outcome (the accumulated value) is the same, and otherwise everything else, too, then because you receive an additional interest in the case of the annuity due, you know that the period payments will be smaller than in the case of an ordinary simple annuity.

Example 3: You owe money to a bank which you pay back at the of each month at 100€ per month. Because your car broke down you miss two payments. At an annual interest rate of 12% compounded monthly, how much do you need to pay back in your third payment to make up for the lost payments?

- ☐ Approach: ordinary simple annuity (as the payment is done at the end of every period and the compound frequency corresponds to the payment frequency).
- ☐ Formula: $S = \frac{(1+i)^n - 1}{i} R$.
- ☐ Knowns: $R = 100$, $n = 3$, $i = 0.12/12$ (careful here as it is the annual interest rate and not the monthly one, i.e. it does not correspond to the payment period). Unknown: S
- ☐ Derivation: $S = \frac{(1+0.12/12)^3 - 1}{0.12/12} 100$. This gives $S = 303.01$.
- ☐ Answer: The total payment in period 3 should be $S = 303.01$€.

General annuity

We said that, in the case of an ordinary simple annuity or an annuity due, the payment frequency corresponds to the compound frequency. This does not need to be the case. If this is not true, then we are in the case of a general annuity.

A general annuity is an annuity where the frequency of compounding does not correspond to the frequency of payment.

Fortunately, we learned how to deal with this already in the previous lecture/section. There we studied which interest rate is equivalent to another interest rate if both have different compound frequencies. So, by using the formula for equivalent interest rates, we can find another interest rate that is equivalent to the one that is given to you, but where we derive the interest rate assuming that the frequency of compounding is equal to the frequency of payment. Once this step is done, then we transformed any general annuity into an ordinary simple annuity (if the periodic payment is at the end of the period) or an annuity due (if the period payment is at the beginning of a period).

The steps to pursue in order to transform a general annuity into an ordinary simple annuity (or annuity due) are as follows. 1) figure out the frequency of repayment; 2) transform the interest rate that is given to you into an equivalent per period interest rate where the new frequency of compounding corresponds to the frequency of repayment. 3) solve it as an ordinary simple annuity (or annuity due).

As an example, if the frequency of repayment is quarterly but the interest is an annual interest that compounds monthly, then you know that you need to find an equivalent per period interest rate that compounds quarterly and is not the annual rate but the quarterly interest rate. Hence you would solve $(1 + j_m/m)^{12} = (1 + i)^4$, where i is your per period interest rate that compounds quarterly.

Example 4: You want to accumulate 10,000€ through bi-annual savings, in July and December, of 200€. If the annually compounded annual interest rate is 5%, then how many payments do you need to make?

- ☐ Approach: general annuity (as the payment is done at the end of every period but the compound frequency does not correspond to the payment frequency).
- ☐ Formula: $S = \frac{(1+i)^n - 1}{i} R$ and $1 + j_1 = (1 + i)^m$.[9]
- ☐ Knowns: $S = 10000$, $j_1 = 0.05$, $R = 200$, $m = 2$. Unknown: n, i
- ☐ Derivation: Using the conversion of interest formula $1 + j_1 = (1 + i)^m$, we get $1 + 0.05 = (1 + i)^2$. Solving for i gives $i = 0.0246951$. This i is the per period interest rate. Then we use the formula for ordinary simple annuities and get $10000 = \frac{(1+0.0246951)^n - 1}{0.0246951} 200$. Simplifying gives $n = \log_{1.0246951}(0.0246951 \cdot 10000/200 + 1)$, giving $n = 32.9628$.

[9]Be careful here: we need the per period interest rate, not the annual interest rate. Remember that $j_m/m = i$.

☐ Answer: We need 33 bi-annual payments, or 16 and a half years.

Example 5: You placed 500€ at the end of each year into a bank account for the past 20 years. However, the interest rate changed. During the first 10 years the interest rate was 5%, while during the crisis that lasted 8 years the rate went down to 1%, and in the last 2 years the interest recovered to 2.5%. How much money is in your bank?

☐ Approach: ordinary simple annuity (as the payment is done at the end of every period and the compound frequency corresponds to the payment frequency).[10] We need to take the changing interest rate into account.

☐ Formula: $S = \frac{(1+i)^n - 1}{i} R$.

☐ Knowns: $R = 500$, $n_1 = 10$, $n_2 = 8$, $n_3 = 2$, $i_1 = 0.05$, $i_2 = 0.01$, $i_3 = 0.025$. Unknown: S

☐ Derivation: We split everything up into 3 periods. $S_1 = \frac{(1+0.05)^{10}-1}{0.05} 500$ is the accumulated value from the first ten years of savings. $S_2 = \frac{(1+0.01)^8-1}{0.01} 500$ is the accumulated value from the eight years of savings during the crisis. $S_3 = \frac{(1+0.025)^2-1}{0.025} 500$ is the accumulated value from the last two years of savings. However, we need to be careful since S_1 is kept in the bank account for another 10 years at an interest of 1% during 8 years and an interest rate of 2.5% during 2 years. Similarly, S_2 is kept in the account for 2 years at a rate of 2.5%. Thus the total accumulated value will be $S = S_1(1 + 0.01)^8(1 + 0.025)^2 + S_2(1 + 0.025)^2 + S_3$. We obtain $S = 12519.84$.

☐ Answer: The total accumulated value will be $S = 12,519.84$€.

(Net) Present Value

If we calculate the value of an annuity, i.e. we use the formula for annuities, then we calculate it at its future value. However, we are sometimes also interested in

[10] Remember that if nothing is written about the periodicity of the interest rate or the compound frequency then you can assume both are annual.

its present value. For example, if you want to compare two annuities that last for different periods of time, then you need to convert these, as always, to the same point in time. One way to do that is by calculating the present value of an annuity, namely the value that an annuity has today, and thus not its future value.

Notation:

- ☐ C - initial payment (cost)
- ☐ P - present value
- ☐ S - accumulated value
- ☐ B - net present value
- ☐ i - per *interest* period interest rate

The **present value of an annuity** is then given by

$$P = \frac{S}{(1+i)^n}.$$

You can immediately see the relationship to our formula for compound interest. In fact, it is the same as the fundamental compound interest formula. Thus, the present value formula is nothing new to you, but it shows you how everything links back to our simple formula on compound interest.

Sometimes we also deal with investments, and these investments come in the form of a down payment that needs to be done at the beginning, but that then for example receives a steady stream of income for some period of time. If you then want to calculate whether an investment is profitable, you need to compare the costs and the benefits from that investment. One way how to do this is by using the so-called **Net Present Value**. This value is obtained if you denote all inflows and outflows in today's value and deduct the outflows from the inflows.

$$\boxed{B = \tfrac{S}{(1+i)^n} - C.} \tag{27}$$

The *Present Value* is the *Net Present Value* without deducting the initial payment. If there were other payments at other points in time then they need to be discounted to today's values, too.

Example 6: Assume you invest 10,000€ into a machine today (in January). You expect a steady stream of earnings from this at each end of the year. You can then place these earnings into a bank at an annually compounded annual interest rate of 3%. At the end of 11 years you believe that the machine will be worthless and won't work any longer. How large do your regular earnings in each period need to be to make this investment profitable?

- ☐ Approach: ordinary simple annuity and Net Present Value.
- ☐ Formula: $S = \frac{(1+i)^n - 1}{i} R$ and $B = \frac{S}{(1+i)^n} - C$.
- ☐ Knowns: $C = 10000$, $n = 11$, $i = 0.03$. We know that $B = 0$ is we want to know the minimum amount of R that is required to make the investment profitable. Unknowns: S, R
- ☐ Derivation: First we substitute the values into the formulas. We have $S = \frac{(1+0.03)^{11} - 1}{0.03} R$ and $0 = \frac{S}{(1+0.03)^{11}} - 10000$. From the second equation we can find S, which will be $S = 13842.3$. Using this in the first formula we can solve for R and get $R = 1080.77$.
- ☐ Answer: The yearly earnings need to be 1,080.77€ in order to break-even.

Internal Rate of Return

Instead of asking what payments or earnings are necessary to make an investment profitable, we can also ask what interest rate is needed to make an investment profitable. This we call the **Internal Rate of Return** and it is the interest rate for which the net present value equals zero.

The Internal Rate of Return is the interest rate at which a project breaks even, i.e. at which the net present value of a project's cash flows equals zero.

Notation:

- ☐ C - initial cost of project
- ☐ R - return per period
- ☐ T - total number of periods
- ☐ i - per period interest rate
- ☐ t - period, for $t = 1, 2, ..., T$.

The cash flows accumulated minus the initial investment can be written as

$$0 = \frac{R_1}{1+i} + \frac{R_2}{(1+i)^2} + ... + \frac{R_T}{(1+i)^T} - C,$$

Then the equation to calculate the internal rate of return looks simple, but it is not so easy to solve for many periods:

$$C = \sum_{t=1}^{T} \frac{R_t}{(1+i)^t} \tag{28}$$

For a limited number of periods we can solve this by resorting to a *trial and error approach*, meaning that we substitute this formula into the calculator with specific guesses for the internal rate of return. We then recursively try to find the interest rate that solves this equation. If our guess yields a number that is negative, then the guess is too high, if it gives a number that is positive, then the guess is too low. We then try a number somewhere in between the low and the high guess and can thus iteratively converge.

Example 7: A machine initially costs 15,000€. After 9 years the machine will become obsolete. The investor expects a return of 2,000€ during the first 6 years, then a return of 1,500€ during the next 2 years, and a return of 1,000 during the last year. What is the internal rate of return on this project?

- ☐ Approach: internal rate of return.
- ☐ Formula: $C = \sum_{t=1}^{T} \frac{R_t}{(1+i)^t}$.

- ☐ Knowns: $R_t = 2000$ for 6 years, then $R_t = 1500$ for 2 years, then $R_t = 1000$ for 1 years, $C = 15000$, $T = 9$. Unknown: i.

- ☐ Derivation: Our formula (neglecting the equal to zero for the moment) then becomes $\frac{2000}{(1+i)} + \frac{2000}{(1+i)^2} + ... + \frac{2000}{(1+i)^6} + \frac{1500}{(1+i)^7} + \frac{1500}{(1+i)^8} + \frac{1000}{(1+i)^9} - P$. Let's start our trial with $i = 0.05$. This gives us a value of -2122.73 That is a negative number, meaning that the initial investment is bigger than the return. Hence we have to use a smaller guess. Let's try $i = 0.01$. This gives a value of 289.6. That value is positive but not very large, so let's try a slightly bigger guess of $i = 0.015$. This yields -47.9266, which implies that the guess was too large. So let's try $i = 0.014$. We then obtain 18.67. Increasing our guess by a little we try $i = 0.0143$, which gives -1.35. We then guess $i = 0.01428$, which gives -0.02. Hence we were able to approximate the internal rate of return to 2 cents, or four digits behind the comma, which is close enough.[11]

- ☐ Answer: The internal rate of return is 1.42%.

One advantage is, however, if the return per period is constant. If $R_t = R$, i.e. constant over time, the equation above simplifies to

$$C = \frac{1 - (1+i)^{-T}}{i} R. \tag{29}$$

What does this formula remind you of? Let's rewrite it slightly by multiplying the top and bottom of the right-hand side by $(1+i)^T$, which gives $P = \frac{(1+i)^T - 1}{i} \frac{R}{(1+i)^T}$. Then, based on our annuity formula, we have $S = \frac{(1+i)^T - 1}{i} R$, and substituting this into the equation gives $C = \frac{S}{(1+i)^T}$. This is simply a special case of the net present value where $B = 0$. Of course, it should be a special case, because the Internal Rate of Return is calculated based on the assumption that the net present value is zero.

[11] A very (though unnecessarily) precise answer would be $i = 0.0142797$.

Full derivation We write out the break even condition, i.e. when the net present value of a project equals zero, where the initial investment is C, the return per period is R_t, the per period interest rate i, and the time horizon is $t = 1, ..., T$.

$$0 = -C + \frac{R_1}{1+i} + \frac{R_2}{(1+i)^2} + ... + \frac{R_T}{(1+i)^T},$$

which equals

$$0 = \sum_{t=1}^{T} \frac{R_t}{(1+i)^t} - C.$$

If now $R_t = R$, $\forall t \geq 1$, then

$$0 = \left(\frac{1}{1+i} + \frac{1}{(1+i)^2} + ... + \frac{1}{(1+i)^T} \right) R - C.$$

This is a geometric progression with common ratio $(1+i)^{-1}$. Given our formula for the sum of a geometric progression[12] we then simply apply the formula and get

$$C = R \frac{(1+i)^{-T} - 1}{(1+i)^{-1} - 1} (1+i)^{-1},$$

which, after simplification, gives us

$$P = R \frac{1 - (1+i)^{-T}}{i}.$$

This formula, of course, is simply the net present value formula where the net present value is equal to zero, combined with the formula for an ordinary simple annuity.

Take-away box

☐ **Ordinary simple annuities**

TAG: *Sequence of payments made at equal intervals in time made at the end of each period. The compounding period corresponds to the periodic payment period.*

- S - accumulated (future) value
- R - periodic payment
- n - number of interest conversion periods during the term of an annuity (= total no. of payments for simple annuities)
- i - per *interest* period interest rate

$$S = \frac{(1+i)^n - 1}{i} R \tag{30}$$

☐ **Annuity due**

TAG: *Sequence of payments made at equal intervals in time (at beginning of period).*

- S - accumulated value
- R - periodic payment
- n - number of interest conversion periods during the term of an annuity (= total no. of payments for simple annuities)
- i - per *interest* period interest rate

$$S = (1+i)\frac{(1+i)^n - 1}{i} R \tag{31}$$

☐ **General annuity**

TAG: *Period payments at frequencies different from the compounding period.*

Calculate the interest rate that corresponds to the period payment frequency and solve as ordinary simple annuity or annuity due (depending whether the payment is at the end or beginning of the period).

☐ **(Net) Present Value**

TAG: *Denoting all inflows and outflows in today's value and deducting the outflows from the inflows.*

- P - initial payment
- C - initial investment cost
- S - accumulated value
- B - net present value
- i - per *interest* period interest rate

The **present value** of an investment is given by

$$P = \frac{S}{(1+i)^n} \tag{32}$$

The **net present value** of an investment is given by

$$B = \frac{S}{(1+i)^n} - C \tag{33}$$

The *Present Value* is the *Net Present Value* without deducting the initial payment. If there were other payments at other points in time then they need to be discounted to today's values, too.

☐ **Internal Rate of Return**

TAG: *The interest rate for which the net present value equals zero.*

- C - initial cost of project
- R - return per period
- T - total number of periods

- i - per period interest rate
- t - period, for $t = 1, 2, ..., T$.

$$C = \sum_{t=1}^{T} \frac{R_t}{(1+i)^t} \tag{34}$$

If $R_t = R$, i.e. constant over time, this simplifies to

$$C = \frac{1-(1+i)^{-T}}{i} R. \tag{35}$$

Lecture 6 - loan amortization

In the previous section we have dealt with regular cash flows that are received, or paid, into an account, and then e.g. compared to an initial investment. What happens, however, if we take out a loan and then repay this loan in regular intervals? For example, imagine you take out a mortgage of 200,000€ and the bank asks you to repay this at regular intervals in time. Then, while interest is charged initially on the original amount of the mortgage, the money that you owe the bank will decrease with each subsequent payment. This implies that your interest payments will get smaller because you owe less money to the bank. While it is simple to calculate this for a short period of time, it gets cumbersome to do this for many periods. Fortunately, mathematics has provided us again with a formula.

Loan amortization applies if we undertake regular payments until a loan is paid off.

Notation:

- ☐ P - principal (loan)
- ☐ R - payment per period
- ☐ X_n - money outstanding
- ☐ N - total number of periods
- ☐ i - per period interest rate
- ☐ n - period, for $n = 1, 2, ..., N$.

Let us derive the formula. Imagine you take out a loan of size P and repay this immediately. Then $P = R$ and $X_0 = 0$. This is an unusual case, so let us assume that you repay the loan fully after one period. In this case you have to pay interest on the loan and need to pay that interest, too, such that $R = P(1+i)$, and $X_1 = 0$. We can solve for P and obtain $P = R/(1+i)$. What generally occurs is, however, that you repay less than the full amount during the first period. Say you only repay $R < P(1+i)$, in which case you would have $X_1 = P(1+i) - R$. If you also do not repay everything in the next period, then the money outstanding on the loan after $n = 2$ will be $X_2 = X_1(1 + i) - R$. Substituting for $X_1 = P(1 + i) - R$, we get $X_2 = P(1+i)^2 - R(1+i) - R$. If we had repaid everything at $n = 2$, then we would have $X_2 = 0$ and solving for P would give us $P = R/(1+i)+R/(1+i)^2$. If you recall our progressions then you see the relationship to the geometric progression from the first lecture. The only difference is that here the common ratio also applies to the first term (because we repay at the end of the first period and thus have to pay interest on this). As a result we can use the formula for the geometric progression but have to multiply with an additional $(1 + i)^{-1}$ term.

Formula for loan amortization

Our formula for loan amortization then becomes

$$P = \frac{1-(1+i)^{-N}}{i} R. \tag{36}$$

- ☐ (**Beware** common mistake:) You should be particular attentive to the exponent on $(1+i)$, as this exponent is $-N$. It is a common mistake to ignore the negative sign here.

- ☐ If you want to use the formula for loan amortization to calculate the initial value of the loan P, then you simply apply the formula as above

$$P = \frac{1 - (1 + i)^{-N}}{i} R.$$

If you want to solve for the size of the regular payments then you would transform the equation to

$$R = \frac{iP}{1 - (1 + i)^{-N}}.$$

If you want to know for how long (in terms of periods) you need to repay a loan, then you solve for N and get

$$N = -\log_{1+i}(1 - iP/R).$$

If you want to solve for the interest rate, then you will have to use our iteration method that we applied for the internal rate of return.

Based on our calculations above, we now also know how to calculate the **debt outstanding**, which is simply

$$X_{n+1} = (1 + i)X_n - R.$$

In words, the amount of debt still outstanding in the next period is given by the amount of debt outstanding in this period (X_n) plus the interest that we have to pay between now and next period (iX_n), and minus the amount of debt that we repay during that period (R).

Example 1: Calculate the monthly payment that is needed in order to repay a loan of 20,000€ after 5 years if the annual interest rate is 7.5% and compounded monthly.

- ☐ Approach: loan amortization and equivalent rates.
- ☐ Formula: loan amortization: $R = iP/(1 - (1 + i)^{-m})$. equivalent rate: $(1 + j_n)^n = (1 + j_m/m)^m$.
- ☐ Knowns: $P = 20000$, $N = 5 \cdot 12 = 60$. Unknown: R and i. For the equivalent rate we have knowns: $j_m = 0.075$, $m = 12$, unknown: i_e.
- ☐ Derivation: We first calculate the per period interest rate i. As the compound frequency is the same between the interest rate that we are given and the period payment, then we can simply calculate the per period interest rate by $i = j_{12}/12 = 0.075/12 = 0.00625$. Substituting everything into the formula we have $R = 0.0062520000/(1 - (1 + 0.00625)^{-60})$, which gives $R = 400.759$.
- ☐ Answer: The monthly payments would be 400.76€.

Example 2: Assume you want to repay a loan of 1,000,000€ on a monthly basis at a monthly compounded, monthly interest rate of 0.2% over 35 years. What are the monthly payments? What would be the annual equivalent rate on this loan?

- ☐ Approach: loan amortization and annual equivalent rates.

☐ Formula: loan amortization: $R = iP/(1-(1+i)^{-m})$. Annual equivalent rate: $1+i_e = (1+j_m/m)^m$.

☐ Knowns: $P = 1000000$, $N = 35 \cdot 12 = 420$, $i = 0.002$. Unknown: R. For the annual equivalent rate we have knowns: $j_m = 12 \cdot 0.002$, $m = 12$, unknown: i_e.

☐ Derivation: Substituting everything into the formula we have $R = 0.0021000000/(1-(1+0.002)^{-420})$, which gives $R = 3521.58$.
The annual equivalent rate would be $i_e = (1+0.002)^{12} - 1$, giving $i_e = 0.0242658$.

☐ Answer: The monthly payments would be $3,521.58$€ and the annual equivalent rate is 2.43%.

Amortization table

An **amortization table** is a table that helps to visualize and track an amortization schedule over time, which includes the periodic payments, the interest expenditure, the principal repaid and the outstanding principal (debt). The only item that we do not know how to calculate yet is the interest expenditure. We do, however, know that we pay interest only on the amount of debt outstanding. As a consequence, we get $I_n = iX_{n-1}$. The regular payments R are obviously the same in every period as they are regular (and constant) payments. The debt outstanding X_n must be decreasing in every period as we are repaying a part of the debt. If our regular payments were too small, then the debt outstanding would be increasing, meaning that every period we pay back less than the per period interest. If a loan is to be paid off, then this can only happen if the debt outstanding also decreases over subsequent periods. In addition, at the final period, if everything is calculated correctly, then the debt outstanding must amount to zero. The interest payments must be decreasing over subsequent periods simply because the interest is calculated based on the debt outstanding, and since the debt outstanding is decreasing then so must be the per period interest payment. Finally, the principal repaid must be increasing in each period as the regular payments are constant but the interest payments decrease over time. Furthermore, the overall sum of the principal repaid must obviously be equal to the principal, which helps as a final checkup whether everything is calculated correctly.

Table 1: Amortization table

n	R	X_n	I_n	$R - I_n$
0	-	$X_0 = P$	-	-
1	R	$X_1 = (1+i)X_0 - R$	iX_0	$R - iX_0$
2	R	$X_2 = (1+i)X_1 - R$	iX_1	$R - iX_1$
$\vdots$				
N	R	$X_T = (1+i)X_{T-1} - R = 0$	iX_{T-1}	$R - iX_{T-1}$
totals	TR	-	$i\sum_0^{T-1} X_t$	$\sum_0^T (R - iX_t) = P$

Example 3: Mister White borrows 2000€ repaid quarterly over two years at an annual interest rate of 24% compounded monthly. Derive an amortization table for his repayment schedule.

- ☐ Approach: loan amortization and equivalent rates.
- ☐ Formula: loan amortization: $R = iP/(1-(1+i)^{-m})$. equivalent rate: $(1 + j_n/n)^n = (1 + j_m/m)^m$.
- ☐ Knowns: $P = 2000$, $N = 2 \cdot 4 = 8$. Unknown: R and i. For the equivalent rate we have knowns: $j_m = 0.24$, $m = 12$, $n = 4$, unknown: j_n.
- ☐ Derivation: We solve first to obtain the per period interest rate by using the equivalent formula, where $(1 + j_n/4)^4 = (1 + 0.24/12)^{12}$. Solving for j_4 we get $j_4 = 0.244832$. However, this is an annual rate and we have to convert it into a per period interest rate. As the periods are quarterly, then we get $i = j_4/4$, giving $i = 0.061208$. using i as well as the other variables in our formula for loan amortization we get the quarterly payments $R = 0.0612082000/(1-(1+0.061208)^{-8})$, which is $R = 323.613$.
- ☐ Answer: Our amortization table then becomes

n	R	X_n	I_n	$R - I_n$
0	-	2000	-	-
1	323.61	1798.81	122.42	201.2
2	323.61	1585.3	110.1	213.51
⋮				
8	323.61	0	18.66	304.89
totals	2588.89		588.89	2000

Debt outstanding

As a final calculation of interest we often want to know much debt is outstanding after a certain period of time. While this can be done via the use of an amortization table, it can become cumbersome if many periods are involved. And again we are lucky, as mathematics comes to the rescue. What a time saver. Remember that our formula for debt outstanding was $X_{t+1} = (1+i)X_t - R$. This equation is something that we call a difference (or recurrence) equation. Fortunately it is linear in X, which helps us in finding a quick solution. The approach is the following.

We solve a difference equation of that kind by first finding a particular solution, and then use this in order to obtain a general one. The particular solution involves an initial guess. Let us assume that $X_n = A$, a constant. Then we use this in our formula for debt outstanding and get $A = (1 + i)A - R$. Solving for A gives $A = R/i$. Then we guess (this comes from experience) that $X_t = B(1 + i)^t + A$. As you see, our guess is that the solution of X_n, for every period n and thus independently of needing to know the X in the previous periods, is of a specific (exponential) form. We already know that $A = R/i$. Then at $n = 0$, we know that $X_0 = P = B(1 + i)^0 + R/i$, which gives $B = P - R/i$. Combining everything gives our **general formula for debt outstanding**

$$X_n = (P - R/i)(1 + i)^n + R/i. \tag{37}$$

this must hold for every period. Let's see whether it works. At $n = 0$ we have $X_0 = (P - R/i)(1+i)^0 + R/i = P$. At $n = 1$ we have $X_1 = P(1+i) - R$, and at $n = 2$ we get $X_2 = (P(1+i) - R)(1+i) - R$. Continuing this up to time $n = N$, we would then have $X_N = P(1+i)^N - \sum_{n=1}^{N} R(1+i)^{n-1}$. We know that $\sum_{n=1}^{N} R(1+i)^{n-1}$ is an ordinary simple annuity. Using the formular for an ordinary simply annuity $S = \frac{(1-i)^N - 1}{i} R$, and then have $X_N = P(1+i)^N - S$. If you recall, in the last period the debt outstanding should be zero, such that $X_N = 0$. Then what we have is $P = \frac{S}{(1+i)^N}$, which is simply the net present value formula when the net present value of an investment is zero. And thus the circle closes.

Example 4: You take out a loan of 200,000€ over 10 years at an annual interest rate of 3% compounded monthly. You repay this loan in monthly installments. Calculate the monthly payments. What is the debt outstanding after 9 years? What is the annual equivalent rate?

- ☐ Approach: loan amortization, debt outstanding, (annual) equivalent rates.
- ☐ Formula: loan amortization: $R = iP/(1-(1+i)^{-m})$. Debt outstanding: $X_n = (P - R/i)(1+i)^n + R/i$. equivalent rate: $(1 + j_n/n)^n = (1 + j_m/m)^m$.
- ☐ Knowns: $P = 200000$, $N = 10 \cdot 12 = 120$. Unknown: R and i. For debt outstanding we want $t = 9 \cdot 12 = 108$. For the equivalent rate we have knowns: $j_n = 0.03$, $m = 12$, $n = 12$, unknown: j_e.
- ☐ Derivation: We solve first to obtain the per period interest rate. In this case it is easy as the annual interest rate is compounded monthly, and the installments are also monthly. Solving for $i = j_{12}/12$ we get $i = 0.0025$. Using i as well as the other variables in our formula for loan amortization we obtain monthly payments of an amount $R = 0.0025200000/(1-(1+0.0025)^{-120})$, which is $R = 1931.21$.
  Then we use our formula for debt outstanding, which is $X_{108} = (200000 - 1931.21/0.0025)(1+0.0025)^{108} + 1931.21/0.0025$, and gives us $X_{108} = 22803$. Finally we use the formula for the annual equivalent rate to get $j_e = (1 + 0.03/12)^{12} - 1 = 0.030416$
- ☐ Answers: The monthly repayments amount to 1,931.21€. After nine years the debt outstanding is 22,803€. And the annual equivalent rate which is equal to an annual interest rate of 3% that is compounded monthly is given by 3.04%.

Take-away box

- ☐ **Loan amortization**

  Tag: *Regular payments until loan is payed off.*

  - P - principal (same as present value in annuity)
  - R - payment per period
  - T - number of payment periods
  - i - per *interest* period interest rate

  $$R = \frac{iP}{1-(1+i)^{-T}} \tag{38}$$

  If we want to calculate the *debt outstanding*, then the formula is

  $$X_{t+1} = X_t(1+i) - R. \tag{39}$$

  One problem is that we need to know X_t to solve for X_{t+1}. However, we can explicitly solve this equation for any period t without needing to know the debt outstanding in the previous period. Then we have to use the equation

  $$X_t = \left(P - \frac{R}{i}\right)(1+i)^t + \frac{R}{i}. \tag{40}$$

- ☐ The **Amortization Table** for loan amortization looks as follows. We have t for time period, X for debt outstanding, R for periodic payment, I for interest payment in that period, $P - I_t$ is principal repaid in period t.

Loan Amortization Table

t	X	R	I	R-I
0	$X_0 = P$	0	$I_0 = 0$	0
1	$X_1 = (P - \frac{R}{i})(1+i) + \frac{R}{i}$	$R = \frac{iP}{1-(1+i)^{-T}}$	$I_1 = iX_0$	$R - I_1$
2	$X_2 = (P - \frac{R}{i})(1+i)^2 + \frac{R}{i}$	$R = \frac{iP}{1-(1+i)^{-T}}$	$I_2 = iX_1$	$R - I_2$
$\vdots$				
T	$X_T = 0$	$R = \frac{iP}{1-(1+i)^{-T}}$	$I_t = iX_{t-1}$	$R - I_T$
Total		TR	$\sum_{t=1}^{T} I_t$	$\sum_{t=1}^{T} R - I_t = X_0$

Some final words

This course lays the foundation for a lot of the mathematical issues that you will see afterwards during your studies. Furthermore, it should also help you later in your professional life as you now should be able to evaluate and compare investment projects that last for some periods of time.

However, you are not done yet. The last hurdle to pass this course consists of completing the final exam well enough. I know some of you suffered, and you will have to suffer a bit more until this is done. So I here want to give you some more advice on how you can maximize your chances of getting a good grade in the final exam. Firstly, you should have done all the exercises that I provided you with, not only once but at least twice. These exercises will prepare you well for the final exam. You should have also been able to complete them quickly and on your own. If you come across a problem that you cannot solve then go back to the lecture notes, discuss with your colleagues, or contact me. Upon receiving the correct answer, be sure you understand it, and then go back to the question some time later to see whether you remember the approach and the steps taken. Unless you are a natural-born talent like Ramanujan, you will need to study hard and do these exercises repeatedly.

A final word of advice: Do not procrastinate. You cannot study and comprehend this course in just a couple of days. I wish you the best of luck for the exam and also for your future studies.